Chapter 6 : Forms & Vector Calculus

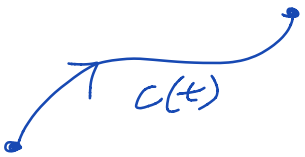
Recall that one of our goals for the course is to generalize the FTOC

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

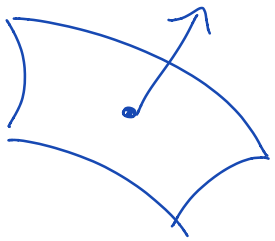
To be able to do that we need several ingredients:

1. orientation (e.g. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$)

e.g.



e.g.



2. differential forms

6.1 Forms on $\mathbb{R}^n$

Def'n: k-Form on $\mathbb{R}^n$ is a function φ that takes k vectors in $\mathbb{R}^n$ & returns a number. φ is multilinear & antisymmetric

Example: The determinant is an n-form on $\mathbb{R}^n$.

Example: k-Form

Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$

Let $V := [\vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_k] \in \mathbb{R}^{n \times k}$

$$= \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{1k} \\ v_{21} & v_{22} & & v_{2k} \\ \vdots & \vdots & \ddots & \\ v_{n1} & v_{n2} & & v_{nk} \end{bmatrix}$$

$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ ($k \times k$)

Takes the matrix formed by the $i_1, i_2, \dots i_k^{th}$ rows of V and returns its determinant

Example

- $dx_1 \wedge dx_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right)$

$$= \det \left(\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \right) = 5 - 8 = -3$$

- $dx_2 \wedge dx_1 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \det \left(\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \right) = 3$

- $dx_1 \wedge dx_2 \wedge dx_4 \left(\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix} \right)$

$$= \det \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 5 \end{bmatrix} \right) = 5$$

- $dx_1 \wedge dx_4 \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) = 0$ (why?)

- Def'n/example:

0-form on $\mathbb{R}^n$ is a function that take no vectors and returns a number

$\Rightarrow$ it is a number

- Remark: only makes sense to talk about k-forms in $\mathbb{R}^n$ when $k \leq n$

(why?)

Meaning of a k-form

Let's start with 2-forms on $\mathbb{R}^3$

$$dx_1 \wedge dx_2 \left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

= signed area of the parallelogram spanned by by $\vec{V}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

$\vec{V}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

Elementary Forms: $dx_{i_1} \wedge \dots \wedge dx_{i_k}$

where $1 \leq i_1 < \dots < i_k \leq n$

Remarks:

- $dx_i \wedge dx_j = -dx_j \wedge dx_i$

- $dx_1 \wedge dx_2 \wedge dx_3 = dx_2 \wedge dx_3 \wedge dx_1$
  $= dx_3 \wedge dx_1 \wedge dx_2$

  elementary form

  (why?)

- In general:

  Suppose $i_1, \dots, i_k$ and $j_1, \dots, j_k$ are the same integers but in a possibly different order.

  Then, there is a permutation $\sigma \in \text{perm}_k$

  $\sigma : j_1 = i_{\sigma(1)}, \dots, j_k = i_{\sigma(k)}$

  and

  $$dx_{j_1} \wedge \dots \wedge dx_{j_k} = \text{sgn}(\sigma)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Some definitions:

Let ϕ & ψ be two k-forms on $\mathbb{R}^n$

We define

$$(\phi+\psi)(\vec{v}_1,\dots,\vec{v}_k) := \phi(\vec{v}_1,\dots,\vec{v}_k)+\psi(\vec{v}_1,\dots,\vec{v}_k)$$

Let a be a scalar. We define

$$(a\phi)(\vec{v}_1,\dots,\vec{v}_k) := a\left(\phi(\vec{v}_1,\dots\vec{v}_k)\right)$$

$A_c^k(\mathbb{R}^n)$:= the space of k-forms on $\mathbb{R}^n$

Theorem: The elementary k-forms form a basis for $A_c^k(\mathbb{R}^n)$:

- $\forall \phi \in A_c^k(\mathbb{R}^n)$, ϕ can be uniquely written as

$$\phi = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Sum over all indices corresponding to elementary forms — coefficient — elementary form

where $a_{i_1 \cdots i_k} = \varphi(\vec{e}_{i_1}, \cdots, \vec{e}_{i_k})$

Example: Let $f_{\vec{v}}(\vec{w}) = \vec{v} \cdot \vec{w}$ ($\vec{v} \in \mathbb{R}^n$, $\vec{w} \in \mathbb{R}^n$)

(Here $\vec{v}$ is fixed)

We can check that $f_{\vec{v}}(\vec{w})$ is a 1-form on $\mathbb{R}^n$ (check this!)

By the above theorem, $f_{\vec{v}}(\vec{w})$ can be written as a linear combination of all the elementary 1-forms

i.e., $f_{\vec{v}} = a_1 dx_1 + \cdots + a_n dx_n$

where $a_i = f_{\vec{v}}(\vec{e}_i) = \vec{v} \cdot \vec{e}_i = v_i$

So $f_{\vec{v}} = v_1 dx_1 + \cdots + v_n dx_n$

and

$$f_{\vec{v}}(\vec{w}) = \sum_{i=1}^{n} v_i \underbrace{dx_i(\vec{w})}_{w_i} = \sum_{i=1}^{n} v_i w_i$$

— x —

Theorem: The dimension of $A_c^k(\mathbb{R}^n)$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proof: The dimension of the space is the

number of basis vectors, i.e., the number of elementary forms.

This is the number of ways you can pick k items from a set of n items.



Examples

$$\dim\left(A_c^1(\mathbb{R}^n)\right) = \binom{n}{1} = n$$

$$\dim\left(A_c^2(\mathbb{R}^3)\right) = \dim\left(A_c^1(\mathbb{R}^3)\right) = 3$$

$$\dim\left(A_c^0(\mathbb{R}^3)\right) = \dim\left(A_c^3(\mathbb{R}^3)\right) = 1$$

— x —

We have defined addition & scalar multiplication for k-forms. We now define the wedge product of forms

Defn: Let $\phi \in A_c^k(\mathbb{R}^n)$, $\omega \in A_c^\ell(\mathbb{R}^n)$.

We define $\phi \wedge \omega \in A_c^{k+\ell}(\mathbb{R}^n)$ by

$$(\phi \wedge \omega)(\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots \vec{v}_{k+\ell})$$

$$= \sum_{\sigma \in S} \operatorname{sgn}(\sigma) \phi(\vec{v}_{\sigma(1)}, \dots, \vec{v}_{\sigma(k)}) \omega(\vec{v}_{\sigma(k+1)}, \dots, \vec{v}_{\sigma(k+\ell)})$$

Where S is the set of permutations with

$$\sigma(1) < \sigma(2) < \dots < \sigma(k)$$

and $\sigma(k+1) < \dots < \sigma(k+\ell)$

wedge product

Example: Wedge product of 1-forms

Let φ & ω be two 1-forms

To figure out $\varphi \wedge \omega$ using the def'n, we should firs figure out the set S that we sum over.

A good way to do this is to list the permutations & decide which belong to S.

$\vec{v}_1 \mid \vec{v}_2$ ($\operatorname{sgn}(\sigma)=1$) & $\vec{v}_2 \mid \vec{v}_1$ ($\operatorname{sgn}(\sigma)=-1$) are the permutations and they both $\in S$.

So $(\varphi \wedge \omega)(\vec{v}_1, \vec{v}_2) = \varphi(\vec{v}_1)\omega(\vec{v}_2) - \varphi(\vec{v}_2)\omega(\vec{v}_1)$

Remark: This is consistant with how we defined $dx_1 \wedge dx_2$:

$$dx_1 \wedge dx_2(\vec{a}, \vec{b}) = \det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)$$

($dx_2(\vec{a})$ $dx_1(\vec{b})$)

$$= a_1 b_2 - a_2 b_1$$

($dx_1(\vec{a})$ $dx_2(\vec{b})$)

Example: $\varphi \in A_c^2(\mathbb{R}^n)$, $\omega \in A_c^1(\mathbb{R}^n)$

$\varphi \wedge \omega = ?$

Soln:

① List the permutations $\in \text{perm}_3$ ($\to 2+1$) and decide which belong to S.

$\vec{v}_1 \vec{v}_2 \mid \vec{v}_3$	$\vec{v}_1 \vec{v}_3 \mid \vec{v}_2$	$\vec{v}_2 \vec{v}_1 \mid \vec{v}_3$
$(1<2\mid 3)$	$1<3\mid 2$	$2 \not< 1$ ✗
$\vec{v}_2 \vec{v}_3 \mid \vec{v}_1$	$\vec{v}_3 \vec{v}_1 \mid \vec{v}_2$	$\vec{v}_3 \vec{v}_2 \mid \vec{v}_1$
$2<3\mid 1$	$3 \not< 1$ ✗	$3 \not< 2$ ✗

② Figure out their signatures

$\vec{v}_1 \vec{v}_2 \mid \vec{v}_3 \to +$

$\vec{v}_1 \vec{v}_3 \mid \vec{v}_2 \to -$ (why?)

$\vec{v}_2 \vec{v}_3 \mid \vec{v}_1 \to +$

③ $(\varphi \wedge \omega)(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

$= \varphi(\vec{v}_1, \vec{v}_2)\omega(\vec{v}_3) - \varphi(\vec{v}_1, \vec{v}_3)\omega(\vec{v}_2)$

$+ \varphi(\vec{v}_2, \vec{v}_3)\omega(\vec{v}_1)$

Exercise: apply this to

$(dx_1 \wedge dx_2) \wedge dx_3\,(\vec{u}, \vec{v}, \vec{w})$

Properties of the wedge product

$\phi \wedge (\omega_1 + \omega_2) = \phi \wedge \omega_1 + \phi \wedge \omega_2$

(distributivity)

$(\phi_1 \wedge \phi_2) \wedge \phi_3 = \phi_1 \wedge (\phi_2 \wedge \phi_3)$

(associativity)

If $\phi \in A_c^k(\mathbb{R}^n)$, $\omega \in A_c^\ell(\mathbb{R}^n)$

$\phi \wedge \omega = (-1)^{k\ell}\, \omega \wedge \phi$

(show commutativity)

Form fields

a form field is a map that takes points in $\mathbb{R}^n$ & returns k-forms.

Formally:

A k-form field on an open subset $U \subset \mathbb{R}^n$ is a map $\varphi : U \to A_c^k(\mathbb{R}^n)$.

The space of all k-form fields is denoted $A^k(U)$

Moreover, since forms can be written as linear combinations of elementary forms we have

$$\varphi(\vec{x}) = \sum_{1 \le i_1 \dots < i_k \le n} \overbrace{a_{i_1 i_2 \dots i_k}(\vec{x})}^{\text{functions of } \vec{x}\ (\mathbb{R}^n \to \mathbb{R})} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Interpretation: At every point $\vec{x}$, there is a form that takes k-dimensional parallelogs and returns numbers (e.g., their volume, mass ...)

Example:

Consider the 2-form field on $\mathbb{R}^3$ given by $\varphi = \cos(xy)\, dx \wedge dz$.

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ & $\vec{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

Calculate $[\varphi(\vec{x})](\vec{v}_1, \vec{v}_2)$

Sol'n: $[\varphi(\vec{x})](\vec{v}_1, \vec{v}_2)$

$$= \cos(xy) \det\left(\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\right)$$

$$= \cos(xy) \cdot 1$$

So we see: at different $\vec{x}$'s

$[\varphi(\vec{x})](\vec{v}_1, \ldots \vec{v}_k)$ can give different answers even for the same $\vec{v}_1, _, \vec{v}_k$.

Notation:

$$\varphi(P_{\vec{x}}(\vec{v}_1 _, \vec{v}_k)) = [\varphi(x)](\vec{v}_1 __, \vec{v}_k)$$

$P_{\vec{x}}$: parallelogram anchored at $\vec{x}$

6.2 Integrating form fields over parametrized domains

Usually, discussing this requires discussing the "orientation of a manifold" (which we will do soon)

Let $\gamma: U \to \mathbb{R}^k$ where $U \subset \mathbb{R}^n$ $\quad \{ \varphi \in A^k(\mathbb{R}^n)$ ↳ form field

Def'n:

$$\int_{[\gamma(U)]} \varphi := \int_U \varphi\Big(P_{\gamma(\vec{u})}\big(\vec{D}_1\gamma(\vec{u}), \dots, \vec{D}_k\gamma(\vec{u})\big)\Big)\,|d^k u|$$

(depends on $\gamma(\vec{u})$)

↳ image of U under γ + orientation

Example: Let $\gamma: [0,a] \to \mathbb{R}^2$, $u \mapsto (R\cos u, R\sin u)$

and let $\varphi = x\,dy - y\,dx$

be a 1-form field on $\mathbb{R}^2$

By definition:

$$\int_{[\gamma(V)]} \overbrace{x\,dy - y\,dx}^{\varphi}$$

$$= \int_{[0,a]} (x\,dy - y\,dx)\left(P_{\begin{pmatrix} R\cos u \\ R\sin u \end{pmatrix}}\left(\overbrace{\begin{pmatrix} -R\sin u \\ R\cos v \end{pmatrix}}^{\vec{D}_1\gamma(u)}\right)\right)|du|$$

$$= \int_{[0,a]} \underbrace{(R\cos u)}_{\det(\leftarrow)}\underbrace{R\cos v}_{\det(\downarrow)} - (R\sin u)(-R\sin u)\,|du|$$

$$= \int_{[0,a]} R^2\,|du| = R^2 a .$$

Example: (2-form field in $\mathbb{R}^3$)

Let $\varphi = dx \wedge dy + y\,dx \wedge dz$

and let $\gamma(s,t) = (s+t, s^2, t^2)$

where $(s,t) \in \underbrace{[0,1]\times[0,1]}_{S}$

By definition:

$$\int_{[\gamma(S)]} \varphi = \int_S (dx\wedge dy + y\,dx\wedge dz)\left(P_{\begin{pmatrix} s+t \\ s^2 \\ t^2 \end{pmatrix}}\left(\begin{bmatrix} 1 \\ 2s \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2t \end{bmatrix}\right)\right) ds\,dt$$

$\frac{d}{ds}\begin{pmatrix} t+s \\ s^2 \\ t^2 \end{pmatrix}$ $\frac{d}{dt}\begin{pmatrix} s+t \\ s^2 \\ t^2 \end{pmatrix}$

$$= \int_0^1\int_0^1 \det\begin{pmatrix} 1 & 1 \\ 2s & 0 \end{pmatrix} + s^2 \det\begin{pmatrix} 1 & 1 \\ 0 & 2t \end{pmatrix} ds\,dt$$

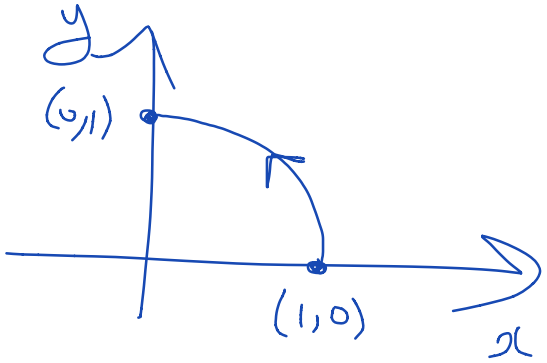
$$= \int_0^1\int_0^1 -2s + s^2(2t)\,ds\,dt = \ldots = -2/3$$

Exercise: Repeat the above calculation with $\gamma(t) = \begin{pmatrix} s+t \\ t^2 \\ s^2 \end{pmatrix}$ which parametrizes the same surface as before.

The answer will be $-2/3$.

One last example to highlight the importance of orientation

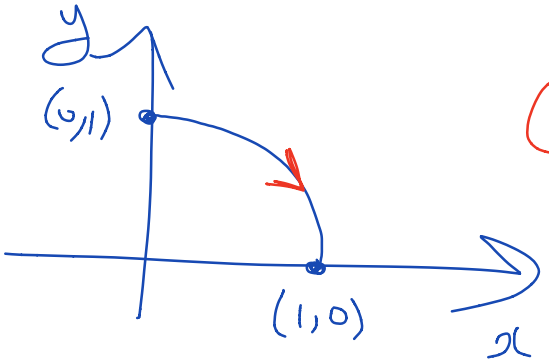
Let $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $0 \le t \le \frac{\pi}{2}$



$$\int_{[\gamma(U)]} dy = \int_{[0,\pi/2]} dy\left(P\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}\right)|dt|$$

$$= \int_0^{\pi/2} \cos t\, dt = +1$$

Now, let $\delta(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ $0 \le t \le \frac{\pi}{2}$



(Same curve opposite orientation)

$$\int_{[\delta(U)]} dy = \int_{[0,\pi/2]} dy\left(P\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}\right)|dt|$$

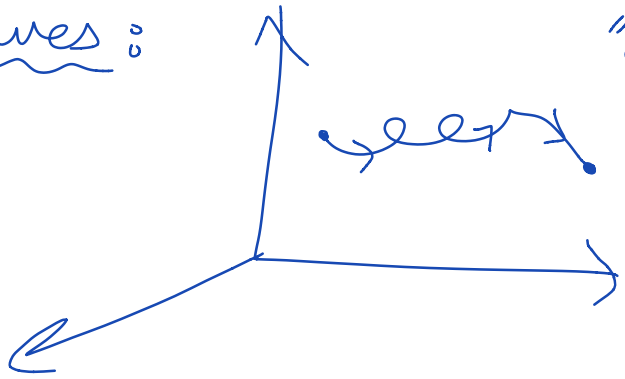
$$= \int_0^{\pi/2} -\sin t\, dt = -1$$

So even though we had the same curve, Parametrizing it differently, gave us different answers (due to the flip in orientation)

6.3) Orientation of Manifolds

Idea:

curves:



"drive along a path in a certain direction"

surfaces:



"water flows through the surface in a certain direction"

More generally: ?

Defn:

Let V be a finite dimensional real vector space, and set $\mathcal{B}_V$ to be the set of all bases of V. An orientation of V is a map

$$\Omega : \mathcal{B}_V \longrightarrow \{+1, -1\}$$

such that if $\{\vec{v}\}$ and $\{\vec{w}\}$ are two bases with change of basis matrix $[P_{w \to v}]$

then

$$\Omega(\{\vec{v}'\}) = \operatorname{sgn}\left(\det\left([P_{v' \to v}]\right)\right) \Omega(\{\vec{v}\}).$$

A basis $\{w\} \in \mathcal{B}_V$ is called direct if $\Omega(w) = 1$ & indirect if $\Omega(w) = -1$

Since

$$\Omega(\{\vec{v}'\}) = \mathrm{sgn}\left(\det\left([P_{v'\to v}]\right)\right)\Omega(\{\vec{v}\}).$$

you can decide in the very beginning that

$$\Omega(\{v'\}) = \pm\,\mathrm{sgn}(\det[P_{v'\to v}])$$

(corresponding to $\Omega(v)=+1$ & -1)

$\Omega^{\{v\}}$ is the orientation for which $\{\vec{v}\}$ (normally chosen to be the standard basis) is direct. & denoted Ω^{st}

Example: Let S be the subspace spanned by the vectors $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ & $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ in $\mathbb{R}^3$

and note that points in S satisfy $x+y+z=0$.

Then $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ & $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ are a basis for S (call it $\{v_1\}$)

& $\begin{pmatrix}0\\1\\-1\end{pmatrix}$, $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ are also a basis for S (call it $\{v_2\}$)

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{V_3} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{V_1} \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}}_{P_{V_1 \to V_3}}$$

$$\det\left(\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$$

$\Rightarrow V_1$ & V_3 define opposite orientations

Important for later

Orienting a manifold

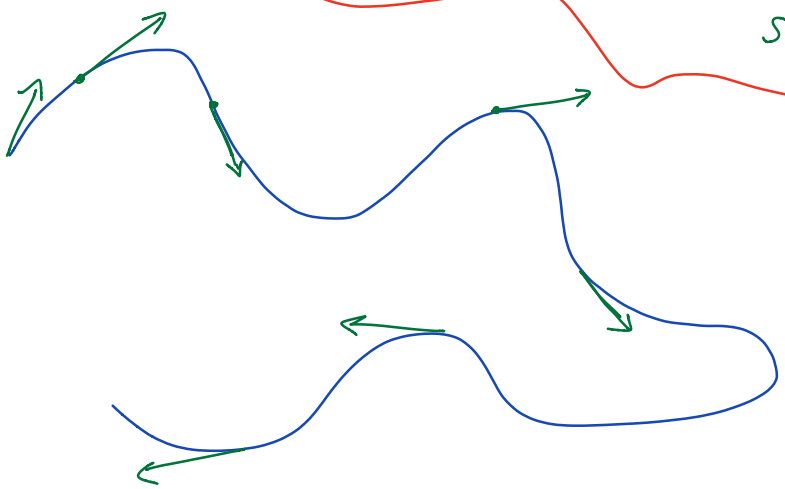
$\{(x, \vec{v}_1, \dots, \vec{v}_k) \in \mathbb{R}^{n \times (k+1)}\}$

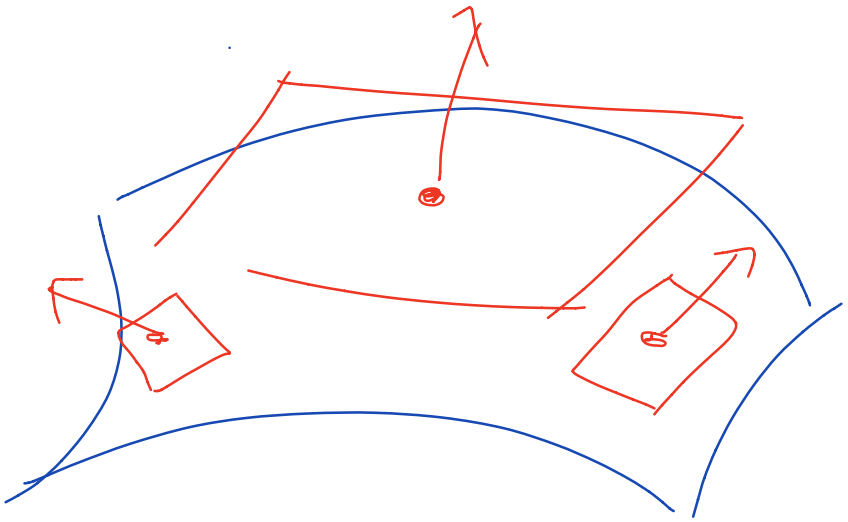
$\vec{v}_1, \dots, \vec{v}_k$: basis for $T_x(M)$

An orientation of a k-dimensional manifold $M \subset \mathbb{R}^n$ is a continuous map $\mathcal{B}(M) = \{+1, -1\}$ whose restriction is an orientation of $T_x(M)$

$T_x(M)$: Tangent space of M

e.g.





Proposition:

(I) Orienting points: $\Omega : \{x\}_{i=1}^{N} \longrightarrow \pm 1$

The tangent space for each pt. is a 0-dim vector space (and 0-dim vector spaces are oriented by $\{\emptyset\} \to +1$ (or -1)). (check this)

(II) Orienting open subsets of $\mathbb{R}^n$

Let $U \subset \mathbb{R}^n$ be open & n-dimensional

Then at every pt. in U $T_x U = \mathbb{R}^n$

& $\Omega_x^{st} = \Omega^{st}$

proof: Clear (why?)

(III) Orienting a curve

Let $C \subset \mathbb{R}^n$ be a smooth curve. A non-vanishing tangent vector field $\vec{t}$ that varies continuously with x defines the orientation

$$\Omega^{\vec{t}}_x(\vec{v}) = \operatorname{sgn}\left(\vec{t}(x) \cdot \vec{v}\right)$$

for every basis $\vec{v}$ of $T_x C$. For $\Omega^{\vec{t}}$ the basis $\vec{t}(x)$ of $T_x C$ is direct for all $\vec{x} \in C$.

Proof: (1) Say $\vec{v}_1$ & $\vec{v}_2$ are 2 bases of $T_x C$ then $\vec{v}_2 = c\vec{v}_1$ for $c \neq 0$.
↳ change of basis "matrix"

(2) Moreover, by definition

$$\operatorname{sign}\left(\vec{t}(x) \cdot \vec{v}_2\right) = \operatorname{sign}\left(\vec{t}(x) \cdot c\vec{v}_1\right)$$

so

$$\underbrace{\operatorname{sign}\left(\vec{t}(x) \cdot \vec{v}_2\right)}_{\Omega(\{\vec{v}_2\})} = \operatorname{sign}\left(\det(c)\right) \underbrace{\operatorname{sign}\left(\vec{t}(x) \cdot \vec{v}_1\right)}_{\Omega(\{\vec{v}_1\})}$$

$\det(c)$: change of basis matrix

(3) Still need to check continuity of $\operatorname{sign}(\vec{E}(x)\cdot\vec{v}_2)$

Note that $(x,\vec{v}) \longrightarrow \vec{E}(x)\cdot\vec{v}$ is continuous on $C \times \mathbb{R}^n$

↳ the curve

and $\vec{E}(x)\cdot\vec{v} \neq 0$ whenever $\vec{v}$ is a basis for $T_x C$

Finally, $\operatorname{sign}(\cdot)$ is continuous on $\mathbb{R}-\{0\}$

So $\Omega^{\vec{E}}$ is cont. on $(x,\vec{v})$ where $\vec{v}$ is a basis of $T_x C$.

(IV) Orienting a surface in $\mathbb{R}^3$

$S \subset \mathbb{R}^3$ be a smooth surface & let

$\vec{n} : S \to \mathbb{R}^3$ be a transverse vector field on S.

$\forall \vec{x} \in S$, $\vec{n}(x) \notin T_x S$,

$\vec{n}(x) \neq 0$, $\vec{n}$ is cont. on S.

Then $\Omega^{\vec{n}}(\vec{v}_1,\vec{v}_2) = \operatorname{sign}\left(\det[\vec{n}(x),\vec{v}_1,\vec{v}_2]\right)$ (bases for $T_x S$)

(This is called the orientation of S by the transverse vector field $\vec{n}$).

Proof: We have to show that

$$\Omega^{\vec{n}}(\vec{v}_1, \vec{v}_2) = \text{sign}\left(\det(\vec{n}(x), \vec{v}_1, \vec{v}_2)\right)$$

Viewed as a function $(\vec{x}, \vec{v}_1, \vec{v}_2) \longrightarrow \text{sign}(d$ — is continuous and its restriction to $T_x(M)$ is an orientation of $T_x(M)$. .

First: $(\vec{x}, \vec{v}_1, \vec{v}_2) \longrightarrow \det\left([\vec{n}(x), \vec{v}_1, \vec{v}_2]\right)$ is continuous. Moreover, $\vec{n} \notin T_x(S)$ & $\neq 0$ and $\vec{v}_1, \vec{v}_2$ span $T_x(S)$ so $\det[\vec{n}(x), \vec{v}_1, \vec{v}_2] \neq 0$ on $\mathcal{B}(S) := \{(\vec{x}, \underbrace{\vec{v}_1, \vec{v}_2}_{\text{basis for } T_x(S)})\}$

$\Rightarrow \Omega^n(\vec{v}_1, \vec{v}_2) = \text{sign}\left(\det(\vec{n}(x), \vec{v}_1, \vec{v}_2)\right)$ is cont.

Second: Suppose $\vec{v}_1, \vec{v}_2$ & $\vec{w}_1, \vec{w}_2$ each span $T_x(S)$ with COB matrix

$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \Rightarrow$ COB matrix for $\vec{n}(x), \vec{v}_1, \vec{v}_2$ & $\vec{n}_x, \vec{w}_1, \vec{w}_2$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & \tilde{P} \end{bmatrix} = \tilde{P}$

so that $\det(\tilde{P}) = \det(P)$

$\Rightarrow$

$$\Omega^n(\vec{v}_1, \vec{v}_2) = \text{sign}\left(\det(\vec{n}(x), \vec{v}_1, \vec{v}_2)\right)$$

$$= \text{sign}\left(\det \tilde{P}\right) \text{sign}\left(\det(\vec{n}(x), \vec{v}_1, \vec{v}_2)\right)$$

$$= \text{sign}\left(\det \tilde{P}\right) \Omega^n(\vec{v}_1, \vec{v}_2)$$ ∎

Example: the curve given by

$(x, y, z) = (\cos\theta, \sin\theta, \theta) \quad \forall \theta > 0$

has $\vec{T}_{(x,y,z)} = (-\sin\theta, \cos\theta, 1) = (-y, x, 1)$

defines a counterclockwise/upward orientation for C (why?)

Example: Let $S \subset \mathbb{R}^3$ be the sphere given by $\underbrace{x^2+y^2+z^2}_{f(x,y,z)} = R^2$ for some $R > 0$

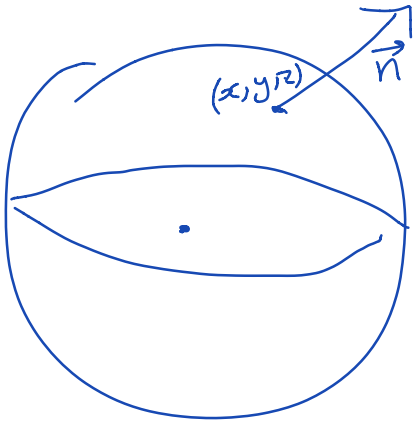
Then $\vec{n}(x,y,z) = (x,y,z)$
defines an orientation of S

To check: note that $\vec{n} \notin T_{\vec{x}}(S)$ because

$$[Df]|_{(x,y,z)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2x \; 2y \; 2z) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(gradient essentially)

$$= 2R^2 \neq 0 \quad \cdots$$ (Recall $\vec{v} \in \ker Df \Rightarrow \vec{v} \in T_{\vec{x}}(S)$)



(orientation by the outward normal)

What about manifolds given by equations

Let $U \subset \mathbb{R}^n$ be open, $F: U \to \mathbb{R}^{n-k}$ be a C^1-map such that $[DF(x)]$ is surjective at all $x \in M := \vec{F}^{-1}(\vec{0})$. Then

(The points in M satisfy $F(\vec{x}) = 0$)

$$\Omega_x : \mathcal{B}(T_x M) \to \{+1, -1\}$$

$$\vec{v}_1, \dots, \vec{v}_k \mapsto \operatorname{sign} \det(\nabla F_1(x), \dots, \nabla F_{n-k}(x), \vec{v}_1, \dots, \vec{v}_k)$$

is an orientation of M.

Example consider the manifold S given by

$$F(\vec{x}) = \sin(x + yz) = 0 \quad \text{(surface in } \mathbb{R}^3\text{)}$$

- $$\nabla F(x) = \begin{bmatrix} \cos(x+yz) \\ z\cos(x+yz) \\ y\cos(x+yz) \end{bmatrix}$$

- $\nabla F(\vec{x})$ orthogonal to S

$\Rightarrow \forall \vec{v} \in T_x S \quad \vec{\nabla} F(x) \cdot \vec{v} = 0$

$\Rightarrow$ transverse

Example: Möbius Strip is non-orientable surface
(proof by paper folding)

Proposition

If M is a connected manifold

(i.e. $\forall u_1, u_2 \in M$ $\exists$ a continuous $\gamma : [a,b] \to M$
$\gamma(a) = u_1, \gamma(b) = u_2$)

then either M is non-orientable or it has two orientations.

In the latter case, specifying an orientation of $T_x M$ at some point defines the orientation everywhere.

Proof: (1) Suppose M is orientable & $\Omega: B(M) \to \{+1, -1\}$ is an orientation. Then $-\Omega$ is also an orientation. (2) Now, we need to show there are no other orientations.

Let Ω' be an orientation such that $\Omega'_{x_0} = \Omega_{x_0}$ for some $x_0 \in M$. We want to show that $\Omega_x = \Omega'_x$ $\forall x \in M$. Let $\gamma: [0,1] \to M$ s.t. $\gamma(0) = x_0$ and $\gamma(1) = x$ (note that $\forall t \in [0,1]$ $\exists\, s(t) \in \{+1, -1\}$ s.t.

$\Omega'_{\gamma(t)} = s(t)\,\Omega_{\gamma(t)}$ and $s(t)$ is continuous (why?) $\Rightarrow s(t) = \underset{\forall t}{+1}$ (or -1)

but $\Omega_{\gamma(0)} = \Omega'_{\gamma(0)} \Rightarrow s = 1$. ∎